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Knots which Behave Like the Prime Numbers

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Accessibility

Knots which behave like the prime numbers

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10 November 2012

Abstract

This paper establishes a version of the Chebotarev density theorem in which number fields are replaced by 3-manifolds.

Contents

1 Introduction

Let $K_1, K_2, \ldots$ be a sequence of disjoint, smooth, oriented knots in a closed, connected 3-manifold M. Let $L_n = \bigcup_1^n K_i$ and let G be a finite group. A surjective homomorphism

$$\rho : \pi_1(M - L_n) \to G$$

determines a covering space $\widetilde{M} \to M$ with Galois group G, possibly ramified over the first n knots. The remaining knots yield a sequence of conjugacy classes $[K_i] \subset G$.

Following Mazur, we say (K_i) obeys the *Chebotarev law* if for any ρ as above and any conjugacy class $C \subset G$, we have

$$\lim_{N\to\infty} \frac{|\{n < i \leq N \, : \, [K_i] = C\}|}{N} = \frac{|C|}{|G|}.$$

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This law is a topological version of the classical Chebotarev theorem (see e.g. [Neu, §13]), with $\widetilde{M} \to M$ playing the role of a field extension and with knots playing the role of primes [Ma2].

Using a result of [PP], we will show:

Theorem 1.1 *Let X be a closed surface of constant negative curvature, and let $K_1, K_2, \ldots \subset M = T_1(X)$ be the closed orbits of the geodesic flow, ordered by length. Then (K_i) obeys the Chebotarev law.*

Theorem 1.2 *The same result holds for the closed orbits (K_i) of any topologically mixing pseudo-Anosov flow on a closed 3-manifold M.*

Examples in fibered manifolds. Let M be a closed 3-manifold which fibers over the circle with pseudo-Anosov monodromy $f : X \to X$. Then the periodic cycles of f determine a sequence of disjoint knots $K_i \subset M$. Suitably ordered, these knots obey the Chebotarev law.

Indeed, f can be regarded as the first return map for a pseudo-Anosov flow on M. A pseudo-Anosov flow is topologically mixing if there are two closed orbits whose lengths satisfy $L(K_i)/L(K_j) \not\in \mathbb{Q}$ (see Corollary 2.2 below). This property can be achieved by making a generic time change, which only affects the ordering of the knots.

Examples in S^3. Let $M \to S^1$ be the torus bundle with Anosov monodromy f corresponding to the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then the complement of the zero section in M is homeomorphic to the complement of the figure–eight knot in S^3 (see e.g. [BZ, p.73]). Since the Chebotarev law persists under Dehn surgery along any of the knots K_i, Theorem 1.2 implies:

Corollary 1.3 *The knots $K_i \subset S^3$ arising from the periodic cycles of monodromy around the figure–eight knot, ordered by their lengths in a generic metric, obey the Chebotarev law.*

If desired, the figure–eight knot itself can be included among the list of knots (K_i). The same construction works for any fibered hyperbolic knot in S^3.

We have included this example because S^3, like $\mathbb{Q}$, admits no unramified extensions. Thus knots in S^3 are analogous to rational primes, and the profinite group $\varprojlim \widehat{\pi}_1(S^3 - L_n)$ is analogous to the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [Ma2].

All these examples are based on the idea that the long closed orbits should wind around each other randomly, at the same time as they become equidistributed in M.

Compact groups. Theorem 1.2 can be generalized to the case where G is a compact Lie group. In this case we require that $\rho : \pi_1(M - L_n) \to G$ has a dense image, and we say the Chebotarev law holds if

$$\frac{1}{N}\sum_{i=n+1}^{N} f([K_i]) \to \int_G f(g)\, dg$$

for any continuous class function $f \in C(G)$.

Let G^0 denote the connected component of the identity for G. In §6 we will show:

Theorem 1.4 *The closed orbits (K_i) of a topologically mixing pseudo-Anosov flow obey the Chebotarev law provided G^0 is semisimple.*

For example, if $G = \mathrm{SU}(2)$, then the values of $(1/2)\,\mathrm{tr}\,\rho(K_i)$ are uniformly distributed with respect to the measure $(2/\pi)\sqrt{1-x^2}\,dx$ on $[-1, 1]$. The same measure arises in the statement of the Sato–Tate conjecture for elliptic curves (see e.g. [Ma1]).

Theorem 1.4 can fail when $G = S^1$, as we will see in §6.

Notes and references. Our treatment emphasizes the connection between symbolic flows and directed graphs. To connect symbolic dynamics to finite branched covers, the most significant points are Lemmas 3.2 and 5.1 below; these ensure that every element of G can be represented by a closed orbit in M. For related work on knots, primes and dynamics, see e.g. [Fra], [Sha], [Ch], [Mor] and the references therein. A special case of Theorem 1.1 (for covers of $T_1(X)$ induced by covers of X) is stated in [Sun, Prop. II-2-12].

I would like to thank B. Mazur for raising the question of the existence of Chebotarev arrangements of knots.

2 Symbolic dynamics

In this section we formulate the Chebotarev theorem in the setting of symbolic dynamics [PP].

Graphs, shifts and flows. Let Γ be a nonempty finite directed graph, with vertices $V(\Gamma)$ and edges $E(\Gamma)$. Assume that each edge $e = (v, w)$ is uniquely determined by its initial and terminal vertices, and that each vertex has both incoming and outgoing edges. The bi-infinite paths in Γ determine a subshift of finite type

$$\Sigma(\Gamma) = \{x : \mathbb{Z} \to V(\Gamma) \ : \ (x_i, x_{i+1}) \in E(\Gamma) \ \text{ for all } i \in \mathbb{Z}\}.$$

Let $\sigma : \Sigma(\Gamma) \to \Sigma(\Gamma)$ be the shift map, given by $\sigma(x)_i = x_{i+1}$.

We will always assume that Γ is *irreducible*. This means the following equivalent conditions hold:

(I.1) Any two vertices in Γ can be joined by a directed path.

(I.2) The graph Γ is connected, and every edge of Γ is part of a directed loop.

(I.3) The shift map $\sigma|\Sigma(\Gamma)$ has a dense orbit.

The suspended flow. Define a metric on $\Sigma(\Gamma)$ by $d(x, x') = \sup_i 2^{-|i|}\delta(x_i, x'_i)$, where $\delta(v, v') = 1$ if $v \neq v'$ and $= 0$ otherwise. Given a Hölder continuous function $h : \Sigma(\Gamma) \to (0, \infty)$, the corresponding *suspended subshift* is defined by

$$\Sigma(\Gamma, h) = \Sigma(\Gamma) \times \mathbb{R} \;/\; \langle (\sigma(x), t) \;\sim\; (x, t + h(x)) \rangle.$$

This space comes equipped with a natural flow, defined by $s \cdot [x, t] = [x, s+t]$ for all $s \in \mathbb{R}$.

Mixing and circle factors. A flow on a space X is *topologically mixing* if for any two nonempty open sets U and V, we have $(t \cdot U) \cap V \neq \emptyset$ for all $t \gg 1$. At the other extreme, a flow has a *circle factor* if there is an $a > 0$ and a continuous map $p : X \to S^1$ such that

$$p(t \cdot x) = e^{iat} p(x) \tag{2.1}$$

for all $t \in \mathbb{R}$.

Principal bundles. Now let $\alpha : \Sigma(\Gamma) \to G$ be a Hölder continuous map from the shift space to a compact Lie group G. (The case of a finite group is allowed.) From this data we obtain a principal G-bundle over the base $\Sigma(\Gamma, h)$; it is given by

$$\Sigma(\Gamma, h, \alpha) = \Sigma(\Gamma) \times \mathbb{R} \times G \;/\; \langle (\sigma(x), t, g) \;\sim\; (x, t + h(x), g\alpha(x)) \rangle.$$

This bundle carries a natural $\mathbb{R}$-action $s \cdot [x, t, g] = [x, s + t, g]$ lifting the flow on the base.

The Chebotarev law. Any closed orbit $\tau \subset \Sigma(\Gamma, h)$ can be lifted to a path in $\Sigma(\Gamma, h, \alpha)$ which connects $[x, 0, \mathrm{id}]$ to $[x, 0, g]$ for some g. The conjugacy class of $[g]$ is independent of the choice of lift, and will be denoted by $[\tau] \subset G$. It represents the holonomy of the G-bundle around τ.

Let $\tau_1, \tau_2, \ldots$ be the closed orbits of $\Sigma(\Gamma, h)$, ordered by length. We say $f \in C(G)$ is a *class function* if $f(gxg^{-1}) = f(x)$ for all $g \in G$. The *Chebotarev law* holds if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f[\tau_i] = \int_G f(x)\, dg(x)$$

for all class functions f. (Here dg is the Haar measure of total mass one.) We may now state:

Theorem 2.1 (Parry–Pollicott) *Suppose the flow on the G-bundle $\Sigma(\Gamma, h, \alpha)$ has a dense orbit and no circle factor. Then the Chebotarev law holds, and the flow is topologically mixing.*

For a proof, see [PP, Theorem 8.5]. (Note: it is implicitly assumed in this reference that $\widetilde{\sigma}_f$ has a dense orbit.)

Corollary 2.2 *If the ratio $L(\tau_i)/L(\tau_j)$ is irrational for some i and j, then $\Sigma(\Gamma, h)$ is topologically mixing.*

Proof. Apply the result above with G the trivial group. If mixing fails then there is a circle factor as in (2.1), which implies $L(\tau_i) \in (2\pi/a)\mathbb{Z}$ for all i. ■

3 Flat bundles and dense orbits

In this section we discuss flat G-bundles over a symbolic flow, and give a condition for the Chebotarev law to hold which only makes reference to dynamics on the base.

Collapsing to a graph. There is a natural continuous projection map

$$C : \Sigma(\Gamma, h) \to \Gamma \tag{3.1}$$

that sends $x \times [0, h(x)]$ linearly to the edge of Γ joining x_0 to x_1. The suspended flow can be thought of as a single-valued resolution of the flow along the directed edges of Γ, which can take several different branches at each vertex.

Chebotarev for flat bundles. Let $\rho : \pi_1(\Gamma) \to G$ be a homomorphism to a compact Lie group G. The map ρ determines a flat principal G-bundle over Γ. Pulling it back by C, we obtain a bundle of the form $\Sigma(\Gamma, h, \widetilde{\rho})$ considered in the preceding section.

Each closed orbit $\tau \subset \Sigma(\Gamma, h)$ projects under C to give a loop in Γ, and hence a conjugacy class in $\pi_1(\Gamma)$. Taking its image under ρ, we obtain the class $[\tau] \subset G$ defined in §2.

Let G^0 denote the connected component of the identity of G. In this section we will show:

Theorem 3.1 *The Chebotarev law holds for the compact G-bundle $\Sigma(\Gamma, h, \widetilde{\rho})$ provided:*

1. *The flow on the base $\Sigma(\Gamma, h)$ is topologically mixing;*
2. *G^0 is semisimple; and*
3. *The image of ρ is dense in G.*

Lemma 3.2 *If the image of ρ is dense in G, then $\Sigma(\Gamma, h, \widetilde{\rho})$ has a dense orbit.*

Proof. Fix a vertex $v \in \Gamma$, and let $S \subset \pi_1(\Gamma, v)$ be the semigroup arising from *directed* loops in Γ, i.e. those which respect the directions of the edges. We claim $\rho(S)$ is dense in G.

As in §2, we assume Γ is irreducible. Given $g \in \operatorname{Im}(\rho)$, let $\tau = (v_0, \ldots, v_n)$ be a loop of adjacent vertices in Γ with $v_0 = v_n = v$ such that $\rho(\tau) = g$. If $(v_i, v_{i+1}) \in E(\Gamma)$ for all i, then τ respects the direction of edges and hence $g \in \rho(S)$.

Now suppose one of the edges is backwards, say $e = (v_i, v_{i+1}) \notin E(\Gamma)$. Then $-e = (v_{i+1}, v_i) \in E(\Gamma)$. By irreducibility of Γ, there is a directed loop $\mu \in \pi_1(\Gamma, v_{i+1})$ that begins with $(-e)$. Now replace e with $e\mu^k$ in τ, and cancel e with $(-e)$. The result is a new loop τ_k based at v, with fewer backward edges. The holonomy for the new loop has the form $\rho(\tau_k) = g_1 h^k g_2$, where $g_1 g_2 = g$. With a suitable choice of $k \gg 0$ we can arrange that h^k is as close to the identity as we wish, and hence $\rho(\tau_k) \approx g$. Repeating this process for each backward edge of τ, we conclude that $g \in \overline{\rho(S)}$, and hence $\overline{\rho(S)} = G$.

The rest of the proof is straightforward: start with a bi-infinite path τ in Γ that encodes a dense flow line for $\Sigma(\Gamma, h)$, and then insert loops, using the fact that $\overline{\rho(S)} = G$, to ensure that its lift to the G-bundle over $\Sigma(\Gamma, h)$ is dense as well. ■

Proof of Theorem 3.1. Suppose $\Sigma(\Gamma, h, \widetilde{\rho})$ has a circle factor, given by a map p to S^1 satisfying $p(t \cdot x) = e^{iat} p(x)$ for some $a > 0$. Since the actions of G and $\mathbb{R}$ commute, the function $p(gx)/p(x)$ is constant along flow lines, and hence globally constant by the Lemma above. Its unique value $\chi(g) = p(gx)/p(x)$ defines a continuous homomorphism $\chi : G \to S^1$. (Cf. [PP, Prop. 8.4].)

Since G^0 is semisimple and G is compact, the image $\chi(G) = \chi(G/G^0) \subset S^1$ is a finite group. Thus $p(x)^n$ is G-invariant for some $n \geq 1$, so it descends to give a circle factor for $\Sigma(\Gamma, h)$, contrary to our assumption that the flow on the base is topologically mixing.

Thus $\Sigma(\Gamma, h, \widetilde{\rho})$ has a dense orbit and no circle factor, so it obeys the Chebotarev law by Theorem 2.1. ∎

4 Markov sections

Let $M = T_1(X)$ be the unit tangent bundle of a closed hyperbolic surface X of genus $g \geq 2$. The time t geodesic flow on M will be denoted by $x \mapsto t \cdot x$. We will refer to a periodic orbit $\gamma \subset M$ as a *closed geodesic*, and denote its length by $L(\gamma)$.

In this section we review the theory of Markov sections and the symbolic encoding of the geodesic flow. For details, see e.g. [Bo], [Ser], [PP] and [Ch].

Rectangles. The manifold M is covered by the unit tangent bundle $T_1(\mathbb{H})$ of the hyperbolic plane, and we have a natural fibration

$$\Delta : T_1(\mathbb{H}) \to S^1 \times S^1 - (\text{diagonal}).$$

The fiber over (a, b) is the unique oriented geodesic which runs from a to b.

A *rectangle* $R \subset T_1(\mathbb{H})$ is the image of a smooth section of Δ over a product of closed intervals $A \times B$. The product structure $R \cong A \times B$ is determined intrinsically by the stable and unstable manifolds of the geodesic flow. Flowing for positive time shrinks the A factor, and expands the B factor. We let ∂R denote the four edges of R, and $\operatorname{int}(R) = R - \partial R$ its interior (an open disk).

By definition, a *rectangle* $R \subset M = T_1(X)$ is a simply-connected set that lifts to a rectangle $\widetilde{R} \subset T_1(\mathbb{H})$.

Consider a finite collection of disjoint rectangles $R_i \cong A_i \times B_i \subset M$. Assume for all $x \in \bigcup R_i$, there is a $t > 0$ such that $t \cdot x \in \bigcup R_i$. The least such t gives the *return time* $r(x)$, and the *first return map* is defined by

$$f(x) = r(x) \cdot x.$$

This function is continuous at x so long as $f(x) \in \bigcup \operatorname{int}(R_i)$. In particular, it is continuous on the locus

$$R_{ij} = \{x \in \operatorname{int}(R_i) \ : \ f(x) \in \operatorname{int}(R_j)\}.$$

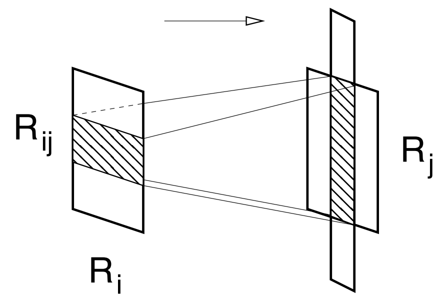

Figure 1. Geodesic flow on a Markov section.

Markov sections. The rectangles (R_i) provide a *Markov section* of the geodesic flow if for all i, j there are open intervals $A_{ij} \subset A_j$ and $B_{ij} \subset B_i$ such that

$$R_{ij} \cong A_i \times B_{ij} \quad \text{and} \quad f(R_{ij}) \cong A_{ij} \times B_j.$$

See Figure 1. As shown in the references cited above,

> *The geodesic flow for a compact surface of negative curvature admits a Markov section.*

Symbolic dynamics. A Markov section $(R_i)_{i=1}^n$ determines a graph Γ with vertices $V(\Gamma) = \{v_i\}_{i=1}^n$ and edges

$$E(\Gamma) = \{e_{ij} = (v_i, v_j) \ : \ R_{ij} \neq \emptyset\}.$$

Since the geodesic flow has a dense orbit, this graph is irreducible.

The sequence of rectangles visited by the orbits of the first return map $f : \bigcup R_i \to \bigcup R_i$ determines a unique Hölder continuous, surjective, *symbolic encoding* map

$$p : \Sigma(\Gamma) \to \bigcup R_i,$$

characterized by the property that that $p(..., x_{-1}, x_0, x_1, \ldots) \in R_i$ if $x_0 = v_i$, and $p(\sigma(x)) = f(p(x))$ if $p(x) \in \bigcup R_{ij}$. There is a unique Hölder continuous height function on $\Sigma(\Gamma)$ such that $h(x) = r(p(x))$ whenever $p(x) \in \bigcup R_{ij}$; taking the suspension, we obtain a continuous, surjective map

$$\pi : \Sigma(\Gamma, h) \to M$$

sending the symbolic flow to the geodesic flow.

Periodic orbits. The symbolic encoding of an orbit $\gamma \subset M$ is unique unless γ passes through the edge of some rectangle. But any two orbits passing through the same edge are asymptotic in forward time or asymptotic in backward time. Thus at most *one* periodic geodesic passes through each edge of the Markov section. It follows easily that the map

$$\tau \mapsto \gamma = \pi(\tau)$$

gives a bijection between closed orbits, satisfying $L(\tau) = L(\gamma)$, once finitely many closed orbits have been excluded from $\Sigma(\Gamma, h)$ and M.

5 A spine for the geodesic flow

Finally we relate the homotopy class of a closed geodesic to its symbolic encoding, and deduce Theorems 1.1 and 1.2.

Let $E_{ij} = \bigcup_{x \in R_{ij}} [0, r(x)] \cdot x$ be the union of the geodesic segments running from R_i to R_j. Then

$$U = \left(\bigcup \operatorname{int}(R_i)\right) \cup \left(\bigcup E_{ij}\right) \tag{5.1}$$

is an open, dense subset of M. It is easy to construct an embedding

$$\iota : \Gamma \to U \subset M$$

such that $p(v_i) \in R_i$ and $p(e_{ij}) \subset R_i \cup E_{ij} \cup R_j$ for all i and j. Since R_i and E_{ij} are contractible, the homotopy class of $\iota : \Gamma \to U$ is uniquely determined by these requirements.

Now consider any closed geodesic $\gamma = \pi(\tau) \subset U$ (all but finitely many closed geodesics have this form). From the definitions above, it follows readily that the maps $\tau \to U$ given by

$$\tau \xrightarrow{\pi} \gamma \subset U \quad \text{and} \quad \tau \xrightarrow{C} \Gamma \xrightarrow{\iota} U \tag{5.2}$$

lie in the same homotopy class. (The projection $C : \Sigma(\Gamma, h) \to \Gamma$ is defined in §3.)

Surjectivity on $\boldsymbol{\pi_1}$. To establish the Chebotarev law for the geodesic flow, it is crucial to show that every conjugacy class in G arises from at least one closed geodesic. This will follow from:

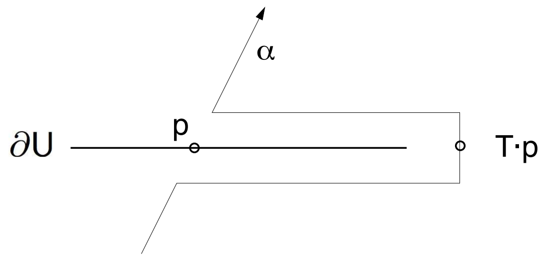

Figure 2. Skirting ∂U.

Lemma 5.1 *Let $L \subset \partial U$ be the union of finitely many closed geodesics. Then the map $\iota : \Gamma \to U \subset (M - L)$ induces a surjective homomorphism*

$$\iota_* : \pi_1(\Gamma) \to \pi_1(M - L).$$

Proof. First assume $L = \emptyset$. The map $\iota : \Gamma \to U$ is a homotopy equivalence, so it suffices to show that $\pi_1(U, x)$ maps onto $\pi_1(M, x)$, where $x \in U$. Equivalently, we will show that a loop $\alpha : S^1 \to M$ based at x can be deformed so its image lies in U.

To see this, first put α into general position with respect to the 2-complex $\partial U = M - U$. Then α crosses ∂U transversely at finitely many points p. By the definition of U, the flowline through p meets $\bigcup \partial R_i$ in forward or backward time. Thus we can assume $p \in [-S, S] \cdot J$, where J is one of the four edges of a rectangle $R_i = A_i \times B_i$.

For concreteness, assume that $J = \{a\} \times B_i$; the case $J = A_i \times \{b\}$ is similar. Then J lies on the unstable manifold of the geodesic flow. Thus after perturbing α slightly, we can assume that the positive geodesic ray through $p = \alpha(s)$ is dense in M. In particular, $T \cdot p \in U$ for some $T > S$.

Now deform the loop $\alpha(t)$ for t near s so that it first approaches p, then shadows the geodesic $[0, T] \cdot p$ through U until it reaches $\alpha(s) = p \cdot T$, and finally returns along nearly the same path, but now on the other side of $[-S, S] \cdot J$. See Figure 2. This deformation reduces the number of intersections between α and ∂U. Thus after finitely many steps we obtain a loop $\alpha : S^1 \to U$, so $\pi_1(U, x)$ maps onto $\pi_1(M, x)$.

To handle the case where $L \neq \emptyset$, we simply start with a loop $\alpha : S^1 \to (M - L)$, and observe that the deformation of α described above is supported in a small neighborhood of $[0, T] \cdot p$. Since the geodesic through p not closed, the interval $[0, T] \cdot p$ is disjoint from L, and thus the deformation can be performed without α crossing L. ∎

Proof of Theorem 1.1. Consider a surjective map $\rho : \pi_1(M - L) \to G$, where G is a finite group and $L = K_1 \cup \cdots \cup K_n$.

Let (R_i) be a Markov section whose rectangles meet L only in their vertices. (To construct (R_i), start with any Markov section for the geodesic flow and subdivide its rectangles horizontally and vertically when they meet L; then apply the geodesic flow for small time, to make these smaller rectangles disjoint from one other.)

Since L is disjoint from $\bigcup \mathrm{int}(R_i)$, the natural embedding of the graph Γ associated to this Markov section is given by a map

$$\iota : \Gamma \to U \subset (M - L).$$

By the Lemma above, the composition

$$\pi_1(\Gamma) \stackrel{\iota_*}{\to} \pi_1(M - L) \stackrel{\rho}{\to} G$$

is surjective. As is well-known, the geodesic flow on M is topologically mixing, so the same is true for the symbolic flow on $\Sigma(\Gamma, h)$. The closed orbits (τ_i) of $\Sigma(\Gamma, h)$ therefore obey the Chebotarev law, by Theorem 3.1.

Now all but finitely many closed geodesic in M have the form $K_i = \pi(\tau_i) \subset U$ with $L(K_i) = L(\tau_i)$. Since the maps in equation (5.2) are homotopic, we have $[K_i] = [\tau_i] \subset G$, and thus the knots (K_i) obey the Chebotarev law as well. ∎

Which loops come from geodesics? Although closed geodesics represent every conjugacy class in G, they do not represent every conjugacy class in $\pi_1(M)$. For example, the fibers of the map $M = T_1(X) \to X$ are not freely homotopic to geodesics.

On the other hand, the proofs of Lemmas 3.2 and 5.1 combine to give an algorithm for constructing a closed geodesic that represents any desired element of G.

6 Pseudo-Anosov flows

The general theory of *pseudo-Anosov flows* on 3-manifolds is discussed in [Mos], [Ca, §6.6] and [Fe]. Examples of pseudo-Anosov flows include the geodesic flows we have just considered, as well as the suspensions of pseudo-Anosov maps on surfaces. A pseudo-Anosov flow need not have a dense orbit [FW], and it may have a circle factor (e.g. in the case of a suspension).

Proof of Theorem 1.4. The proof of Theorem 1.1 used only two properties of the geodesic flow: (i) topological mixing and (ii) the existence of a Markov section. Property (ii) is well-known to hold for pseudo-Anosov flows: see e.g. [Bo] or [PP, App. III] for case of Anosov flows, and [FLP, Exp. 10] for the case of pseudo-Anosov maps. Thus the Chebotarev law holds for any pseudo-Anosov flow that also satisfies (i). Theorem 3.1 shows we need only assume that G is compact and G^0 is semisimple. ∎

Failure of equidistribution on the circle. The Chebotarev law generally fails when $G = S^1$.

For a concrete example in our setting, let $f : X \to X$ be a pseudo-Anosov map on a closed surface of genus $g \geq 2$ which acts trivially on $H_1(X, \mathbb{Z})$. Then the suspension of f gives a pseudo-Anosov vector field v on a 3-manifold M with $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g+1}$. We may assume the corresponding fibration $p : M \to S^1 = \mathbb{R}/\mathbb{Z}$ satisfies $dp(v) = 1$.

Choose two closed orbits of the flow on M such that $[\tau_1]$ and $[\tau_2]$ are linearly independent in $H_1(M, \mathbb{Q})$. (The existence of such orbits follows from the Chebotarev law for $H_1(M, \mathbb{Z}/2)$.) Choose a closed 1-form α on M close to dp, such that $\alpha(v) > 0$ but

$$\phi(\tau_1)/\phi(\tau_2) \notin \mathbb{Q}, \tag{6.1}$$

where $\phi(C) = \int_C \alpha$. Now rescale v so $\alpha(v) = 1$. Then v generates a pseudo-Anosov flow on M such that $L(\tau) = \phi([\tau])$ for all closed orbits τ; in particular, v is topologically mixing by (6.1).

Define $\rho : \pi_1(M) \to S^1$ by $\rho(\gamma) = \phi(\gamma) \bmod 1$. Then the values of $\rho([\tau]) = L(\tau) \bmod 1$ coming from orbits with $L(\tau) \leq M$ are *not* uniformly distributed on S^1. Instead, they tend to concentrate near $M \bmod 1$, since there are exponentially more long orbits than short ones. For more details on this phenomenon, see [PP, pp. 134–136].

Thus the Chebotarev law is broken in this example.

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